

# Generalized Hooke Law for Relativistic Membranes and p-branes

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## Abstract

The character of elastic forces of relativistic membranes and  $p$ -branes encoded in their nonlinear equations is studied. The toroidal brane equations are reduced to the classical equations of anharmonic elastic media described by monomial potentials. Integrability of the equations is discussed and some of their exact solutions are constructed.

## 1 Introduction

A.I. Akhiezer paid much attention to search for effects connected with elastic wave propagation in condensed matter physics [1]. Relativistic membranes ( $p=2$ ) and  $p$ -branes in higher dimensional space-time are fundamental objects of string theory [2], and their macroscopic physics is also controlled by effective elastic forces of fluxes of elementary particle fields, like QCD tubes in string theory. However, quantization of branes is blocked up by nonlinearity of their equations (see e.g. [3-15] and refs. there). The classical and quantum problems of the brane physics deserve great attention and stimulate investigation of the elastic forces associated with relativistic branes.

Here we search the physics of closed  $p$ -branes (with  $p = 2, 3, \dots, (D - 1)/2$ ) evolving in  $D = (2p + 1)$ -dimensional Minkowski space, and find their exact solutions. The brane shape is chosen to be invariant under the global symmetry  $O(2) \times O(2) \times \dots \times O(2)$ . The  $p$ -brane equations are reduced to nonlinear ones of an anharmonic elastic medium with a symmetric stress tensor generated by the interaction Hamiltonian proportional to a

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monomial potential of the degree  $2p$ . Exact solvability of degenerate  $p$ -brane shaped as  $p$ -torus with equal radii, is established. The found solutions are presented by (hyper)elliptic functions that describe  $p$ -branes contracting during the time defined by their energy density and the dimension  $p$ .

## 2 $p$ -branes as elastic media

In the orthogonal gauge  $(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0$ , with the index  $r$  numerating the space-like  $p$ -brane parameters  $\sigma^r$  ( $r = 1, 2, \dots, p$ ), the equations of  $p$ -brane in  $D$ -dimensional Minkowski space are transformed into the second-order PDE for its  $(D - 1)$ -dimensional Euclidean vector  $\vec{x}$  (see [16-18] and refs. there)

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0} \partial_r \left( \frac{T}{\mathcal{P}_0} |g| g^{rs} \partial_s \vec{x} \right), \quad \dot{\mathcal{P}}_0 = 0, \quad (1)$$

where  $\dot{\vec{x}} \equiv \partial_t \vec{x}$ , the energy density  $\mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}}$  with  $g = \det(g_{rs})$ , the induced metric  $g_{rs} = \partial_r \vec{x} \cdot \partial_s \vec{x}$  on the  $p$ -brane hypersurface  $\Sigma_p$ , and the brane tension  $T$ . The system (1) is rather complicated and its general solution is not known. Thus, to get an information on brane physics one can study particular solutions of (1). To find such solutions for branes with a fixed dimension  $p$  ( $p = 2, 3, \dots, (D - 1)/2$ ) we fix the Minkowski space dimension to be odd  $D = 2p + 1$ . Moreover, we suppose that the closed brane hypersurface  $\Sigma_p$  is invariant under the global symmetry  $O(2) \times O(2) \times \dots \times O(2)$ . Then, using the residual gauge symmetry of the orthonormal gauge

$$\tilde{t} = t, \quad \tilde{\sigma}^r = f^r(\sigma^s) \quad (2)$$

we present the Euclidean  $p$ -brane vector  $\vec{x}(t, \sigma^r)$  as

$$\vec{x}^T = (q_1 \cos \sigma^1, q_1 \sin \sigma^1, \dots, q_p \cos \sigma^p, q_p \sin \sigma^p), \quad (3)$$

using the polar coordinate pairs  $(q_r(t), \sigma^r)$ . It results in the diagonalized metric  $g_{rs}(t)$  independent of  $\sigma^r$

$$g_{rs}(t) = q_r^2(t) \delta_{rs}, \quad g = (q_1 q_2 \dots q_p)^2. \quad (4)$$

The anzats (3) shows that the coordinates  $\mathbf{q}(t) = (q_1, q_2, \dots, q_p)$  are the time-dependent radii  $\mathbf{R}(t) = (R_1, R_2, \dots, R_p)$  of the flat  $p$ -torus  $\Sigma_p$ . As a consequence, the energy density  $\mathcal{P}_0$  becomes independent of the  $p$ -torus parameters  $\sigma^r$  and reduces to a constant  $C$  chosen to be positive

$$\mathcal{P}_0 \equiv T \sqrt{\frac{(q_1 q_2 \dots q_p)^2}{1 - \dot{\mathbf{q}}^2}} = C. \quad (5)$$

It means that the Hamiltonian density  $\mathcal{H}_0$  corresponding to  $\vec{x}$  (3) equals the constant  $C$

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2(q_1 q_2 \dots q_p)^2}, \quad (6)$$

where  $\boldsymbol{\pi}(t) = (\pi_1, \pi_1, \dots, \pi_p)$  is the canonical momentum conjugate to  $\mathbf{q}(t)$ . Then Eqs. (1) are reduced to the PDE equations for the world vector  $\vec{x}(t, \sigma^r)$

$$\ddot{\vec{x}} - \left(\frac{T}{C}\right)^2 g g^{rs} \partial_{rs} \vec{x} = 0 \quad (7)$$

which are equivalent to the algorithmic chain of  $p$  nonlinear equations for the components  $q_1, q_2, \dots, q_p$

$$\ddot{q}_r + \left(\frac{T}{C}\right)^2 (q_1 \dots q_{r-1} q_{r+1} \dots q_p)^2 q_r = 0, \quad (8)$$

where the component index  $r$  runs from 1 to  $p$ .

The first integral of the system (8) is given by the relation (5) presented in the form

$$\dot{\mathbf{q}}^2 + \left(\frac{T}{C}\right)^2 (q_1 q_2 \dots q_p)^2 = 1. \quad (9)$$

Eqs. (8) are presented in a compact form as

$$C \ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}, \quad (10)$$

with the elastic energy density  $V(\mathbf{q})$  proportional to the determinant  $g$  of the metric tensor of  $\Sigma_p$

$$V(\mathbf{q}) = \frac{T^2}{2C} g \equiv \frac{T^2}{2C} (q_1 \dots q_p)^2. \quad (11)$$

The equations of an elastic nonrelativistic medium with the mass density  $\rho$  have the form [19]

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}, \quad (12)$$

where  $\ddot{u}_i$  and  $\sigma_{ik}$  are the medium acceleration and the stress tensor, respectively. Then, one can see that Eqs. (10) are presented in the form of Eqs. (12)

$$C \ddot{q}_r = -\frac{T^2}{2C} \delta_{rs} \frac{\partial g}{\partial q_s} \quad (13)$$

with the symmetric stress tensor  $\sigma_{rs}$  defined as

$$\sigma_{rs} = -p \delta_{rs}, \quad p = \frac{T^2}{2C} g \equiv \frac{T^2}{2C} \prod_{s=1}^p q_s^2. \quad (14)$$

The relations (14) show that  $p = V$ ,  $p$  is an isotropic pressure per unit (hyper)area of the  $p$ -brane (hyper)volume, and the constant  $C$  is a relativistic generalization of the rest mass density  $\rho$ . The pressure  $p$  is created by the elastic force  $F_r$

$$F_r = -\frac{\partial V}{\partial q_r} \equiv -\frac{T^2}{C} (q_1 \dots q_{r-1} q_{r+1} \dots q_p)^2 q_r. \quad (15)$$

Relation (15) yields an anharmonic generalization of the Hooke law for the toroidal  $p$ -brane elasticity.

Taking into account that in the totally fixed gauge the Hamiltonian density (6) reduces to the constant  $C$ , one can introduce a new Hamiltonian density

$$\mathcal{H} = \frac{\mathcal{H}_0^2}{2C} = C/2$$

quadratic in the brane momentum  $\boldsymbol{\pi}$ . The brane Hamiltonian  $H$  associated with the density  $\mathcal{H}$  is

$$H = \int d^p \sigma \mathcal{H}, \quad \mathcal{H} = \frac{1}{2C}(\boldsymbol{\pi}^2 + T^2(q_1 \dots q_p)^2). \quad (16)$$

As a result, Eqs. (8) are presented in the Hamiltonian form with the standard PBs

$$\{\pi_a, q_b\} = \delta_{ab}, \quad \{q_a, q_b\} = 0, \quad \{\pi_a, \pi_b\} = 0.$$

The family of the Hamiltonians (16) contains the potential energy terms quartic in  $\mathbf{q}$  for membranes ( $p = 2$ ) and higher monomials for  $p > 2$ , respectively.

The anharmonic Hooke force (15) implemented with the conservation law (9)

$$\sqrt{1 - \dot{\mathbf{q}}^2} = \frac{T}{C}|q_1 q_2 \dots q_p|, \quad (17)$$

restricts the character of the  $p$ -brane motion by

$$0 \leq |\dot{\mathbf{q}}| \leq 1, \quad 0 \leq \frac{T}{C}|q_1 q_2 \dots q_p| \leq 1. \quad (18)$$

The inequalities (18) imply that the velocity value  $|\dot{\mathbf{q}}|$  grows when the  $p$ -brane (hyper) volume  $\sim |q_1 q_2 \dots q_p|$  diminishes, and reaches the velocity of light ( $|\dot{\mathbf{q}}| = 1$ ) while the (hyper)volume vanishes. On the contrary, the minimal velocity  $\dot{\mathbf{q}} = 0$  corresponds to the maximal (hyper)volume  $\sim |q_1 q_2 \dots q_p|$  equal to  $C/T$ .

For the case  $T = 0$ , associated with the tensionless  $p$ -branes [9, 11], Eqs. (8) take the linear form

$$\ddot{\mathbf{q}} = 0, \quad |\dot{\mathbf{q}}| = 1 \quad (19)$$

similar to the equation of free massless particle in the effective space formed by the  $p$ -torus radii.

For the tension  $T$  different from zero the system (8) is rather complicated, and its general solution is unknown. However, one can observe a case when equations (8) may be exactly solved.

### 3 On integrability of $p$ -brane equations

Here we show the exact solvability of the nonlinear  $p$ -brane equations (8) for a degenerate case when all the components of  $\mathbf{q}$  are equal:  $q_1 = q_2 = \dots = q_p \equiv q$ . In this case the system (8) is reduced to the nonlinear differential equation

$$\ddot{q} + \left(\frac{T}{C}\right)^2 q^{(2p-1)} = 0 \quad (20)$$

which integration results in the first integral

$$p\dot{q}^2 + \left(\frac{T}{C}\right)^2 q^{2p} = 1. \quad (21)$$

After the change of  $q$  by  $y \equiv \Omega^{\frac{1}{p}} \sqrt{p} q$ , with  $\Omega \equiv \frac{T}{C} p^{-\frac{p}{2}}$ , Eq. (21) takes the form

$$\left(\frac{dy}{d\tilde{t}}\right)^2 = \frac{1}{2}(1 - y^p)(1 + y^p) \quad (22)$$

with the new time variable  $\tilde{t} \equiv \sqrt{2}\Omega^{\frac{1}{p}}t$ .

For the degenerate toroidal membranes ( $p = 2$ ) Eq. (22) coincides with the canonical equation defining the Jacobi elliptic cosine  $cn(x; k)$

$$\left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2), \quad (23)$$

with the elliptic modulus  $k = \frac{1}{\sqrt{2}}$ .

Thus, the general solution of (23) is

$$y(t) = cn(\sqrt{2\omega}t; \frac{1}{\sqrt{2}})$$

with  $2\omega = T/C$ . After using the relation  $q \equiv y/\sqrt{2\omega}$  we obtain the general solution for the coordinate  $q(t)$

$$q(t) = \sqrt{\frac{C}{T}} cn\left(\sqrt{\frac{T}{C}}(t + t_0); \frac{1}{\sqrt{2}}\right). \quad (24)$$

This solution is similar to the elliptic one earlier obtained in [17, 18] and describing the  $U(1)$  invariant membrane in the five-dimensional (i.e.  $D = 5$ ) Minkowski space. If the initial velocity  $\dot{q}(t_0) > 0$ , the solution (24) describes an expanding torus which reaches the maximal size  $q_{max} = \sqrt{\frac{C}{T}}$  at some moment  $t$ , and then contracts to a point after the finite time  $\mathbf{K}(1/\sqrt{2})\sqrt{\frac{C}{T}}$  (where  $\mathbf{K}(1/\sqrt{2}) = 1.8451$ ) is the quarter period of elliptic cosine).

An explicit equation of the surface  $\Sigma_2(t)$  of the contracting torus (24) is

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= \frac{4C}{T} cn\left(\sqrt{\frac{T}{C}}(t + t_0), \frac{1}{\sqrt{2}}\right)^2, \\ x_1 x_4 &= x_2 x_3. \end{aligned} \quad (25)$$

For the case  $p > 2$  integration of Eq. (22) results in the solution

$$\tilde{t} = \pm \sqrt{2} \int \frac{dy}{\sqrt{1 - y^{2p}}} + \text{const} \quad (26)$$

that contains hyperelliptic integral and defines implicit dependence of  $q$  on the time. Thus, the general solution of Eq. (21) is expressed in terms of hyperelliptic functions generalizing elliptic functions.

The variable change  $z = y^{2p}$  transforms the solution (26) into the integral

$$\tilde{t} - \tilde{t}_0 = \pm \frac{1}{\sqrt{2p}} \int_0^{z^{\frac{1}{2p}}} dz z^{(\frac{1}{2p}-1)} (1 - z)^{-\frac{1}{2}} \quad (27)$$

similar to the integral discussed in [20].

The use of the representation (27) allows to find the contraction time  $\Delta \tilde{t}_c$  of the degenerate  $p$ -torus from its maximal size  $q_{max} = (\frac{C}{T})^{\frac{1}{p}}$  to  $q_{min} = 0$ .

This time turns out to be proportional to the well-known integral

$$\Delta \tilde{t}_c = \frac{1}{\sqrt{2p}} \int_0^1 dz z^{(\frac{1}{2p}-1)} (1 - z)^{\frac{1}{2}-1} = \frac{1}{\sqrt{2p}} B\left(\frac{1}{2p}, \frac{1}{2}\right)$$

which defines the Euler beta function  $B(\frac{1}{2p}, \frac{1}{2})$ .

Coming back to the original time  $t$  and taking into account that  $C = 2E$  we obtain

$$\Delta t_c \equiv \frac{1}{\sqrt{2}} \Omega^{-\frac{1}{p}} \Delta \tilde{t}_c = \frac{1}{2\sqrt{p}} \left(\frac{2E}{T}\right)^{\frac{1}{p}} B\left(\frac{1}{2p}, \frac{1}{2}\right). \quad (28)$$

The representation (28) gives the explicit dependence of the contraction time in term sof of the  $p$ -brane dimension  $p$  and its energy density  $E$ .

So, we conclude that the case of the degenerate toroidal  $p$ -branes with coinciding radii is exactly solvable and connects the solutions of the  $p$ -brane equations with (hyper)elliptic functions.

- A special class of relativistic  $p$ -branes embedded in the  $D = (2p + 1)$ -dimensional Minkowski space is introduced for studying the elastic forces associated with branes. The compact hypersurfaces of these  $p$ -branes are chosen to be invariant under the transformations of the  $O(2) \times O(2) \times \dots \times O(2)$  subgroup of the rotations of  $2p$ -dimensional Euclidean space.
- The brane equations are found to be reduced to equations of the anharmonic elastic media subjected to isotropic pressure dependent of time. Their Hamiltonians including monomial potentials of the degree  $2p$  ( $p = 2, 3, \dots, (D - 1)/2$ ) and yielding nonlinear Hooke forces are constructed.

- The  $p$ -brane equations are proved to be exactly solvable if the brane shapes are similar to  $p$ -tori with equal radii. The constructed (hyper)elliptic solutions describe contracting  $p$ -tori. The exact formula for the contraction time of the toric  $p$ -branes is derived. In particular, these results give the new information on the nonlinear elastic potentials associated with five-branes ( $p = 5$ ) of mysterious M/string theory supposed to exist in the space-time with the exclusive dimension  $D = 11$ .

In particular, these results give the new information on the nonlinear elastic potentials associated with five-branes ( $p = 5$ ) of mysterious M/string theory supposed to exist in the space-time with the exclusive dimension  $D = 11$ . Interestingly, a breakdown of the linear Hooke elasticity and its replacement by a nonlinear anharmonic law, similar to the ones revealed by us, were earlier discovered in 2d and 3d smectics  $\mathcal{A}$  (see e.g. [21], [22]).

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